Effective Lagrangian in nonlinear electrodynamics and its properties of causality and unitarity

Anatoly E. Shabad[1] and Vladimir V. Usov[2]

[1] *P.N. Lebedev Physics Institute, Moscow 117924, Russia*

[2] *Center for Astrophysics, Weizmann Institute, Rehovot 76100, Israel*

Abstract

In nonlinear electrodynamics, by implementing the causality principle as the requirement that the group velocity of elementary excitations over a background field should not exceed the speed of light in the vacuum $c = 1$, and the unitarity principle as the requirement that the residue of the propagator should be nonnegative, we establish the positive convexity of the effective Lagrangian on the class of constant fields, also the positivity of all characteristic dielectric and magnetic permittivity constants that are derivatives of the effective Lagrangian with respect to the field invariants. Violation of the general principles by the one-loop approximation in QED at exponentially large magnetic field is analyzed resulting in complex energy ghosts that signal the instability of the magnetized vacuum. Superluminal excitations (tachyons) appear, too, but for the magnetic field exceeding its instability threshold. Also other popular Lagrangians are tested to establish that the ones leading to spontaneous vacuum magnetization possess wrong convexity.

PACS numbers: 14.70.Bh, 11.55.Fv, 41.20.Jb, 12.20.Ds

I. INTRODUCTION

The effective action that is defined as the Legendre transform of the generating functional of the Green functions [1] and, in its turn, is itself a generating functional of the (one-particle-irreducible) vertices makes a basic quantity in quantum field theory. This is a c-numerical functional of fields and their derivatives, a knowledge of which is meant to supply one with the final solution to the theory. For this reason it seems important to see, how the most fundamental principles manifest themselves as some general properties of the effective action to be respected by model- or approximation-dependent calculations, and whose violation might signal important inconsistencies in the theory underlying these calculations. Such inconsistencies may show themselves first of all as ghosts and tachyons, that play an important role [2] in cosmological speculations about forming the Λ-term and dark energy using a scalar (Higgs) field yet to be discovered in the coming experiments on the Large Hadronic Collider.

It is stated [1] basing on a formal continual integral representation for the propagator that, when the effective action $\Gamma(\phi)$ of a scalar field with mass m is considered, its second variational derivative $\Sigma(x-y|\phi_0) = \delta^2\Gamma/\delta\phi(x)\delta\phi(y)|_{\phi=\phi_0}$ calculated at the constant background value of this field, $\phi(x) = \phi_0$, *i.e.* the mass operator against this background, is a nonpositive quantity, $\Sigma \leq 0$. In other words, the effective Lagrangian is expected – to the extent that this formal property survives perturbative or other approximate calculations – to be a concave = negatively convex function (while the effective potential to be a (positively) convex function) of a constant scalar field. On the other hand, the same statement may be considered as the one directly prescribed by the causality principle. Indeed, the spectral curve of small excitations over the constant field background, $k_0 = \sqrt{\mathbf{k}^2 + m^2 - \Sigma(k)}$, where $k = (k_0, \mathbf{k})$ is the (4-momentum) variable, Fourier-conjugate to the 4-coordinate difference $x - y$, satisfies the causal propagation condition reading that its group velocity should not exceed unity, the absolute speed limit for any signal, $|\partial k_0/\partial\mathbf{k}| = |\mathbf{k}|/k_0 \leq 1$ for any nonnegative mass squared $m^2 \geq 0$, provided, again, that $\Sigma \leq 0$.

The case under our consideration here is much less trivial as we deal not with a massive scalar, but with a massless vector gauge field. The results apply, first of all, to nonlinear electrodynamics, but also to (Abelian sector of) nonAbelian theory. (Nonlinear electrodynamic models, the same as scalar ones, are also considered for cosmological purposes [3] with

the advantage that instead of the scalar field, uncertain to be physically identified, only well established electromagnetic field is involved.)

We are going to demonstrate that the requirement of the causal propagation of elementary excitations over the vacuum occupied by a background field with a constant and homogeneous field strength, supplemented by the requirements of translation-, Lorentz-, gauge-, P- and C- invariances and unitarity has a direct impact on the effective Lagrangian. For the case – which is general for electromagnetic field, but special for a nonabelian field – where the Lagrangian depends on gauge-invariant combinations (field strengthes) $F_{\alpha\beta}(z) = \partial_\alpha A_\beta(z) - \partial_\beta A_\alpha(z)$ of the background field potentials $A_\alpha(z)$, we make sure that the above requirements are expressed as certain inequalities to be obeyed by the first and second derivatives of the effective Lagrangian with respect to the two *field invariants* $\mathfrak{F} = \frac{1}{4}F_{\rho\sigma}F_{\rho\sigma} = \frac{1}{2}(B^2 - E^2)$ and $\mathfrak{G} = \frac{1}{4}F_{\rho\sigma}\tilde{F}_{\rho\sigma} = (\mathbf{EB})$, where $\mathbf{E}$ and $\mathbf{B}$ are background electric and magnetic fields, respectively, and the dual field tensor is defined as $\tilde{F}_{\rho\sigma} = \frac{1}{2}\epsilon_{\rho\sigma\lambda\varkappa}F_{\lambda\varkappa}$, where the completely antisymmetric unit tensor is defined in such a way that $\epsilon_{1230} = 1$. More specifically, we demonstrate that it is a convex function with respect to the both variables $\mathfrak{F}, \mathfrak{G}$ for any constant value of $\mathfrak{F} \gtrless 0$ and $\mathfrak{G} = 0$. Note, the opposite sign of convexity as compared to the scalar field mentioned above.

In Section II model- and approximation-independent study is undertaken.

In Subsection A we remind the general diagonal representation of the polarization operator and photon Green function in terms of its eigenvectors and eigenvalues, obtained for arbitrary values of the momentum k and for nonzero constant field invariants $\mathfrak{F}, \mathfrak{G}$ in [4], and refer to our previous work [5] where limitations on the location of dispersion curves, imposed by demanding that the group velocity of the vacuum excitations be less than/or equal to unity were established for the general case of nonvanishing invariants $\mathfrak{F}$ and $\mathfrak{G}$.

The unitarity requirement that the residue of the Green function in the pole, corresponding to the mass shell of the elementary excitation, be nonnegative (completeness of the set of states with nonnegative norm), is formulated.

In Subsection B we confine ourselves to the infrared asymptotic behavior $k_\mu \to 0$ of the polarization operator, in which case its eigenvalues can be expressed in terms of first and second derivatives of the effective Lagrangian with respect to the field invariants $\mathfrak{F}, \mathfrak{G}$ when these are coordinate-independent. Massless dispersion curves are explicitly found in terms of these derivatives for the "magnetic-like" case $\mathfrak{F} > 0$, $\mathfrak{G} = 0$. The restrictions of Subsection A,

now supplemented with the unitarity requirement, actualize as a number of inequalities, to be satisfied by these derivatives. They mean, in particular, that the effective Lagrangian is a (positively) convex function of the field invariants in the point $\mathfrak{G} = 0$. We reveal the physical sense of the quantities subject to these inequalities as dielectric and magnetic permeabilities responsible for polarizing small static charges and currents of special configurations (There is no universal linear response function able to cover every configuration, which is typical of an anisotropic medium, to which class the magnetized vacuum belongs). In Subsection C the inequalities of Subsection B are extended to include also the "electriclike" background field $\mathfrak{F} < 0$, $\mathfrak{G} = 0$, so in the end the whole axis of the variable $\mathfrak{F}$ is included into result.

In Subsection D we write the (quadratic in the photon field) contribution of the polarization operator into effective Lagrangian, which is local in the infrared limit and presents the Lagrangian for small, slow, long-wave perturbations of the background field (infrared photons). This enables to define their energy-momentum tensor via the Noether theorem. Once this is done, it becomes possible to derive inequalities on the derivatives of the effective Lagrangian basing on alternative pair of general requirements, namely, the Weak Energy Condition and Dominant Energy Condition of Hawking and Ellis [6] that are positivity of the energy density and non-spacelikeness of the energy-momentum flux vector. We demonstrate that within our context the Dominant Energy Condition is equivalent to restrictedness of the group velocity, while the two alternative conditions together lead to a set of inequalities, to which the derivatives of the effective Lagrangian are subjected, that do not contradict to the ones deduced in Subsection B, but cannot be reduced to them. This implies that the Weak Energy Condition is weaker than the positiveness of the residue of the photon propagator exploited in Subsection B.

In Section III we test whether the properties resulting from the general principles as derived in Section II are obeyed within certain approximations and models. First we study the Euler-Heisenberg one-loop effective Lagrangian of Quantum Electrodynamics (Subsection A) and the Lagrangian of Born and Infeld (Subsection B) to establish that the latter perfectly satisfies all of the above properties. On the contrary, due to the lack of asymptotic freedom in QED, some of them are violated by the Euler-Heisenberg Lagrangian at exponentially large magnetic field of Planck scale, leading to appearance of ghosts, signifying the instability of the magnetized vacuum. Superluminal excitations (tachyons) might appear, too, but for the magnetic field exceeding its instability threshold. It is a surprise that

the positive convexity property itself is not violated at any value of the magnetic field. In Subsection C we inspect two one-loop Lagrangians that are known to produce spontaneous magnetic fields. One of them [7] relates to the Yang-Mills theory taken against the uniform background formed by a constant chromomagnetic field directed along a single isotopic direction. The other [8] is a one-loop Lagrangian of electromagnetic field in interaction with a complex massless scalar field taken in de Sitter space. We find that in the both cases the spontaneous magnetization of the vacuum is due to the violation of the positivity property of the Lagrangian convexity, prescribed by the general principles of unitarity and causality. It is notable, however, that in the Yang-Mills case the general properties of the effective Lagrangian established in Section II other than the convexity are well respected by the one-loop approximation, so neither ghosts, nor tachyons appear. We associate this fact with the asymptotic freedom of the underlying theory. In Subsection D another Yang-Mills theory [10], [9] in a constant homogeneous background is inspected, wherein the external field is this time supported by nonzero classical sources and hence a special quantization procedure was used to substitute for gauge invariance.

In the concluding Section IV we attempt a comparative discussion of our approach with other ways of introducing causality into consideration.

II. GENERALITIES

A. Arbitrary dispersion $k_0 \neq 0$, $\mathbf{k} \neq 0$

Let $\mathfrak{L}(z)$ be the nonlinear part of the effective Lagrangian as a function of the two electromagnetic field invariants $\mathfrak{F}$ and $\mathfrak{G}$ and, generally, of other Lorentz scalars that can be formed by the electromagnetic field tensor $F_{\mu\nu}$ and its space-time derivatives. The total action is $S_{\rm tot} = \int L_{\rm tot}(z)\mathrm{d}^4z$, where $L_{\rm tot}(z) = -\mathfrak{F}(z) + \mathfrak{L}(z)$. Once $-\mathfrak{F}$ is the classical Lagrangian the correspondence principle implies that

$$\left.\frac{\delta\Gamma}{\delta\mathfrak{F}}\right|_{\mathfrak{F}=\mathfrak{G}=0} = 0, \tag{1}$$

where $\Gamma = \int \mathfrak{L}(z)\mathrm{d}^4z$.

We consider the background field, which is constant in time and space and has only one nonvanishing invariant: $\mathfrak{F} \neq 0, \mathfrak{G} = 0$ (although $\mathfrak{G}$ may be involved in intermediate

equations). This field is purely magnetic in a special Lorentz frame, if $\mathfrak{F} > 0$, and purely electric in the opposite case, $\mathfrak{F} < 0$. Such fields will be called magnetic- or electric-like, respectively.

Polarization operator is responsible for small perturbations above the constant-field background. In accordance with the role of the effective action as the generating functional of vertex functions, the polarization operator is defined as the second variational derivative with respect to the vector potentials A_μ

$$\Pi_{\mu\tau}(x,y) = \left.\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\tau(y)}\right|_{\mathfrak{G}=0,\mathfrak{F}=\mathrm{const}}. \tag{2}$$

The action Γ here is meant to be - prior to the two differentiations over A_μ, A_τ - a functional containing field derivatives of arbitrary order, but the fields are set constant after the differentiations. Nevertheless, their derivatives do contribute into the polarization operator (2) leading to its complicated dependence on the momentum k, the variable, Fourier conjugated to $(x-y)$.

It follows from the translation- Lorentz-, gauge-, P- and charge-invariance [4, 11, 12] that the Fourier transform of the tensor (2) is diagonal

$$\Pi_{\mu\tau}(k,p) = \delta(k-p)\Pi_{\mu\tau}(k), \qquad \Pi_{\mu\tau}(k) = \sum_{a=1}^{3} \varkappa_a(k)\,\frac{\flat_\mu^{(a)}\ \flat_\tau^{(a)}}{(\flat^{(a)})^2} \tag{3}$$

in the following basis:

$$\flat_\mu^{(1)} = (F^2k)_\mu k^2 - k_\mu(kF^2k), \quad \flat_\mu^{(2)} = (\tilde{F}k)_\mu, \quad \flat_\mu^{(3)} = (Fk)_\mu, \quad \flat_\mu^{(4)} = k_\mu, \tag{4}$$

where $(\tilde{F}k)_\mu \equiv \tilde{F}_{\mu\tau}k_\tau$, $(Fk)_\mu \equiv F_{\mu\tau}k_\tau$, $(F^2k)_\mu \equiv F^2_{\mu\tau}k_\tau$, $kF^2k \equiv k_\mu F^2_{\mu\tau}k_\tau$, formed by the eigenvectors of the polarization operator

$$\Pi_{\mu\tau}\ \flat_\tau^{(a)} = \varkappa_a(k)\ \flat_\mu^{(a)}. \tag{5}$$

We are working in Euclidian metrics with the results analytically continued to Minkowsky space, hence we do not distinguish between co- and contravariant indices. All eigenvectors are mutually orthogonal, $\flat_\mu^{(a)}\flat_\mu^{(b)} \sim \delta_{ab}$, this means that the first three ones are 4-transversal, $\flat_\mu^{(a)}k_\mu = 0$; correspondingly $\varkappa_4 = 0$ as a consequence of the 4-transversality of the polarization operator. The unit matrix is decomposed as

$$\delta_{\mu\tau} = \sum_{a=1}^{4}\frac{\flat_\mu^{(a)}\ \flat_\tau^{(a)}}{(\flat^{(a)})^2} \qquad \text{or} \qquad \delta_{\mu\tau} - \frac{k_\mu k_\tau}{k^2} = \sum_{a=1}^{3}\frac{\flat_\mu^{(a)}\ \flat_\tau^{(a)}}{(\flat^{(a)})^2}. \tag{6}$$

The eigenvalues $\varkappa_a(k)$ of the polarization operator are scalars and depend on $\mathfrak{F}$ and on any two of the three momentum-containing Lorentz invariants $k^2 = \mathbf{k}^2 - k_0^2$, kF^2k, $k\tilde{F}^2k$, subject to one relation $\frac{k\tilde{F}^2k}{2\mathfrak{F}} - k^2 = \frac{kF^2k}{2\mathfrak{F}}$. The squares of the eigenvectors are

$$(\flat^{(1)})^2 = -k^2(kF^2k)((kF^2k) + 2\mathfrak{F}k^2) = k^2k_\perp^2(2\mathfrak{F})^2(k_3^2 - k_0^2),$$
$$(\flat^{(2)})^2 = -(k\widetilde{F}^2k), \quad (\flat^{(3)})^2 = -(kF^2k) \tag{7}$$

The diagonal representation of the photon Green function as an exact solution to the Schwinger-Dyson equation with the polarization operator (3) taken for the kernel is (up to arbitrary longitudinal part):

$$D_{\mu\tau}(k) = \sum_{a=1}^{4} D_a(k)\, \frac{\flat_\mu^{(a)}\ \flat_\tau^{(a)}}{(\flat^{(a)})^2},$$
$$D_a(k) = \begin{cases} (k^2 - \varkappa_a(k))^{-1}, & a = 1, 2, 3 \\ \text{arbitrary}, & a = 4 \end{cases}. \tag{8}$$

The dispersion equations that define the mass shells of the three eigen-modes are

$$\varkappa_a(k^2, \frac{kF^2k}{2\mathfrak{F}}, \mathfrak{F}) = k^2, \qquad a = 1, 2, 3. \tag{9}$$

All the equations above are valid both for magnetic- and electric-like cases, $\mathfrak{F} \lessgtr 0$, $\mathfrak{G} = 0$. If, specifically, the magnetic-like background field $\mathfrak{F} > 0$, $\mathfrak{G} = 0$ is considered, in the special frame the field-containing invariants become

$$\frac{k\tilde{F}^2k}{2\mathfrak{F}} = k_3^2 - k_0^2, \qquad \frac{kF^2k}{2\mathfrak{F}} = -k_\perp^2, \qquad \mathfrak{F} = \frac{B^2}{2}, \tag{10}$$

where we directed the magnetic field $\mathbf{B}$ along the axis 3, and the two-dimensional vector $\mathbf{k}_\perp$ is the photon momentum projection onto the plane orthogonal to it. On the contrary, if we deal with the electric-like background field $\mathfrak{F} < 0$, $\mathfrak{G} = 0$, in the special frame, where only electric field $\mathbf{E}$ exists and is directed along axis 3, we have, instead of (10), the following relations for the background-field- and momentum-containing invariants

$$\frac{k\tilde{F}^2k}{2\mathfrak{F}} = k_\perp^2, \qquad \frac{kF^2k}{2\mathfrak{F}} = k_0^2 - k_3^2, \qquad \mathfrak{F} = \frac{-E^2}{2}, \tag{11}$$

where the two-dimensional vector $\mathbf{k}_\perp$ now is the photon momentum projection onto the plane orthogonal to $\mathbf{E}$. In the both cases the dispersion equations (9) can be represented in the same form

$$\varkappa_a(k^2, k_\perp^2, \mathfrak{F}) = k^2, \qquad a = 1, 2, 3 \tag{12}$$

and their solutions have the following general structure, provided by relativistic invariance

$$k_0^2 = k_3^2 + f_a(k_\perp^2), \qquad a = 1, 2, 3. \tag{13}$$

It is notable that the structure (13) retains [5] when the second invariant is also nonzero, $\mathfrak{G} \neq 0$, this time the direction 3 being the common direction of the background electric and magnetic fields in the special reference frame, where these are mutually parallel.

The causal propagation requires that the modulus of the group velocity, calculated on each mass shell (13), be less or equal to the speed of light in the free vacuum $c = 1$:

$$|\mathbf{v}_{\mathrm{gr}}|^2 = \left(\frac{\partial k_0}{\partial k_3}\right)^2 + \left|\frac{\partial k_0}{\partial \mathbf{k}_\perp}\right|^2 = \frac{k_3^2}{k_0^2} + \left|\frac{\mathbf{k}_\perp}{k_0} \cdot f_a'\right|^2 = \frac{k_3^2 + k_\perp^2 \cdot (f_a')^2}{k_3^2 + f_a(k_\perp^2)} \leq 1, \tag{14}$$

where $f_a' = \mathrm{d}f_a(k_\perp^2)/\mathrm{d}k_\perp^2$. This imposes the obligatory condition on the form and location of the dispersion curves (13), i.e. on the function $f_a(k_\perp^2)$, to be fulfilled within every reasonable approximation (remind that $k_3^2 + f_a(k_\perp^2) \geq 0$ due to (13)) :

$$k_\perp^2 \left(\frac{\mathrm{d}f_a(k_\perp^2)}{\mathrm{d}k_\perp^2}\right)^2 \leq f_a(k_\perp^2). \tag{15}$$

The admissible disposition of dispersion curves was considered by us for the general case of $\mathfrak{G} \neq 0$ in detail in [5]. We found that the massless branches of these curves ("photons"), whose existence is always guarantied by the gauge invariance, for every polarization mode are outside the light cone (or on it) in the momentum space, $k^2 = 0$, whereas the massive branches all should pass below a certain curve in the plane $(k_0^2 - k_3^2, k_\perp^2)$, where k_3 and $\mathbf{k}_\perp$ are the excitation momentum components along and across the direction of the background magnetic and electric fields in the special frame, where these are mutually parallel. We also discussed in that reference why and to what extent the restriction on the group velocity may be equivalent to causality.

Now we proceed by imposing the condition, to be referred to, as unitarity, that the residues of the photon propagator (8) in the poles corresponding to every photon mass shell (9) be nonnegative - the positive definiteness of the norm of every elementary excitation of the vacuum. This requirement implies:

$$1 - \left.\frac{\partial \varkappa_a(k^2, k_\perp^2, \mathfrak{F})}{\partial k^2}\right|_{k_0^2 - k_3^2 = f_a(k_\perp^2)} \geq 0. \tag{16}$$

In the next subsection we shall consider the consequences of requirements (15) and (16) as these manifest themselves in the properties of the effective Lagrangian in the infrared limit.

B. Infrared limit: properties of the Lagrangian as a function of constant fields

Hitherto, we were dealing with the elementary excitation of arbitrary 4-momentum k_μ. To get the (infrared) behavior of the polarization operator at $k_\mu \sim 0$ it is sufficient to have at one's disposal the effective Lagrangian as a function of constant field strengthes, since their space- and time-derivatives, if included in the Lagrangian, would supply extra powers of the momentum k in the expression (2) for the polarization operator. Our goal is to establish some inequalities imposed on the derivatives of the effective Lagrangian $\mathfrak{L}$ over the constant fields by the requirement (15) that any elementary excitation of the vacuum should not propagate with the group velocity larger than unity and the requirement (16) that the residue of the Green function be positive in the photon pole. To proceed beyond this limit we had to include the space and time derivatives of the fields into the Lagrangian. Then, utilizing the same requirements (15), (16) the results concerning the convexity of the effective Lagrangian with respect to the constant fields to be obtained below, might be, perhaps, extended to convexities with respect to the derivative-containing field variables.

Aiming at the infrared limit we do not include time- and space-derivatives of the field strengthes in the equations that follow. Using the definition $F_{\alpha\beta}(z) = \partial_\alpha A_\beta(z) - \partial_\beta A_\alpha(z)$ we find

$$\frac{\delta}{\delta A_\mu(x)} \int \mathfrak{F}(z)\mathrm{d}^4 z = \int F_{\alpha\mu}(z)\frac{\partial}{\partial z_\alpha}\delta^4(x-z)\mathrm{d}^4 z,$$
$$\frac{\delta}{\delta A_\mu(x)} \int \mathfrak{G}(z)\mathrm{d}^4 z = \int \tilde{F}_{\alpha\mu}(z)\frac{\partial}{\partial z_\alpha}\delta^4(x-z)\mathrm{d}^4 z. \tag{17}$$

Then, for the first variational derivative of the action one has

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = \int \left[\frac{\partial\mathfrak{L}(\mathfrak{F}(z),\mathfrak{G}(z))}{\partial\mathfrak{F}(z)}F_{\alpha\mu}(z) + \frac{\partial\mathfrak{L}(\mathfrak{F}(z),\mathfrak{G}(z))}{\partial\mathfrak{G}(z)}\tilde{F}_{\alpha\mu}(z)\right]\frac{\partial}{\partial z_\alpha}\delta^4(x-z)\mathrm{d}^4 z. \tag{18}$$

By repeatedly applying eq. (18) we get for the infrared (IR) limit of the polarization operator in a constant external field

$$\Pi^{\mathrm{IR}}_{\mu\tau}(x,y) = \left.\frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\tau(y)}\right|_{\mathfrak{F},\mathfrak{G}=\mathrm{const}} = \left\{\frac{\partial\mathfrak{L}(\mathfrak{F}(z),\mathfrak{G}(z))}{\partial\mathfrak{F}(z)}\left(\frac{\partial^2}{\partial x_\tau\partial x_\mu} - \Box\delta_{\mu\tau}\right) - \right.$$

$$- \frac{\partial^2\mathfrak{L}(\mathfrak{F}(z),\mathfrak{G}(z))}{\partial(\mathfrak{F}(z))^2}\left(F_{\alpha\mu}\frac{\partial}{\partial x_\alpha}\right)\left(F_{\beta\tau}\frac{\partial}{\partial x_\beta}\right) - \frac{\partial^2\mathfrak{L}(\mathfrak{F}(z),\mathfrak{G}(z))}{\partial(\mathfrak{G}(z))^2}\left(\tilde{F}_{\alpha\mu}\frac{\partial}{\partial x_\alpha}\right)\left(\tilde{F}_{\beta\tau}\frac{\partial}{\partial x_\beta}\right) -$$

$$\left. - \frac{\partial^2\mathfrak{L}(\mathfrak{F}(z),\mathfrak{G}(z))}{\partial\mathfrak{F}(z)\partial\mathfrak{G}(z)}\left[\left(F_{\alpha\mu}\frac{\partial}{\partial x_\alpha}\right)\left(\tilde{F}_{\beta\tau}\frac{\partial}{\partial x_\beta}\right) + \left(\tilde{F}_{\alpha\mu}\frac{\partial}{\partial x_\alpha}\right)\left(F_{\beta\tau}\frac{\partial}{\partial x_\beta}\right)\right]\right\}_{F=\mathrm{const}}\delta^4(x-y). \tag{19}$$

The P-invariance requires that the effective Lagrangian should be an even function of the pseudoscalar $\mathfrak{G}$. Hence the contribution of the last term in eq. (19) – the one in front of the square bracket – vanishes for the "single-invariant" fields with $\mathfrak{G} = 0$ under consideration.

Thus, we find for the infrared limit of the polarization operator in the magnetic- or electric-like field in the momentum representation, $\Pi^{\rm IR}_{\mu\tau}(k,p) = \delta(k-p)\Pi^{\rm IR}_{\mu\tau}(k)$,

$$\Pi^{\rm IR}_{\mu\tau}(k) = \left(\frac{{\rm d}\mathfrak{L}(\mathfrak{F},0)}{{\rm d}\mathfrak{F}}(\delta_{\mu\tau}k^2 - k_\mu k_\tau) + \frac{{\rm d}^2\mathfrak{L}(\mathfrak{F},0)}{{\rm d}\mathfrak{F}^2}(F_{\mu\alpha}k_\alpha)(F_{\tau\beta}k_\beta) + \right.$$
$$\left. + \left.\frac{\partial^2\mathfrak{L}(\mathfrak{F},\mathfrak{G})}{\partial\mathfrak{G}^2}\right|_{\mathfrak{G}=0}(\tilde{F}_{\mu\alpha}k_\alpha)(\tilde{F}_{\tau\beta}k_\beta)\right). \tag{20}$$

Here the scalar $\mathfrak{F}$ and the tensors $F, \tilde{F}$ are already set to be space- and time-independent. By comparing this with (3) we identify the eigenvalues of the polarization operator in the infrared limit as

$$\varkappa_1(k^2, kF^2k, \mathfrak{F})\big|_{k\to 0} = k^2\frac{{\rm d}\mathfrak{L}(\mathfrak{F},0)}{{\rm d}\mathfrak{F}},$$

$$\varkappa_2(k^2, kF^2k, \mathfrak{F})\big|_{k\to 0} = k^2\frac{{\rm d}\mathfrak{L}(\mathfrak{F},0))}{{\rm d}\mathfrak{F}} - (k\tilde{F}^2k)\left.\frac{\partial^2\mathfrak{L}(\mathfrak{F},\mathfrak{G})}{\partial\mathfrak{G}^2}\right|_{\mathfrak{G}=0},$$

$$\varkappa_3(k^2, kF^2k, \mathfrak{F})\big|_{k\to 0} = k^2\frac{{\rm d}\mathfrak{L}(\mathfrak{F},0)}{{\rm d}\mathfrak{F}} - (kF^2k)\frac{{\rm d}^2\mathfrak{L}(\mathfrak{F},0)}{{\rm d}\mathfrak{F}^2}. \tag{21}$$

This is the leading behavior of the polarization operator in the magnetic-like field near zero-momentum point $k_\mu = 0$. Every eigenvalue $\varkappa_a$ is a linear function of $k_\perp^2$ and of $k_0^2 - k_3^2$, hence $\varkappa_a(0,0,\mathfrak{F}) = 0$ for every $a = 1,2,3$. This is a nondispersive approximation, since the refraction index (squared) n_a^2 defined for photons of each mode a on the mass shell (13) as

$$n_a^2 \equiv \frac{|\mathbf{k}|^2}{k_0^2} = 1 + \frac{k_\perp^2 - f_a(k_\perp^2)}{k_0^2} \tag{22}$$

is frequency- and momentum-independent in the infrared limit under consideration.

For the sake of completeness, we give the same eqs. (21) also in terms of the invariant variables

$$\mathcal{B} = \sqrt{\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}} \qquad \mathcal{E} = \sqrt{-\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}} \tag{23}$$

that are, respectively, the magnetic and electric fields in the Lorentz frame, where these are

parallel. Then, with the notation $\widetilde{\mathfrak{L}}(\mathcal{B},\mathcal{E}) = \mathfrak{L}(\mathfrak{F},\mathfrak{G})$ the coefficients in (21) are :

$$\frac{\mathrm{d}\mathfrak{L}(\mathfrak{F},0)}{\mathrm{d}\mathfrak{F}} = \frac{1}{\mathcal{B}}\frac{\mathrm{d}\widetilde{\mathfrak{L}}(\mathcal{B},0)}{\mathrm{d}\mathcal{B}},$$
$$\frac{\mathrm{d}^2\mathfrak{L}(\mathfrak{F},0)}{\mathrm{d}\mathfrak{F}^2} = \frac{1}{2\mathfrak{F}}\left(\frac{\mathrm{d}^2\widetilde{\mathfrak{L}}(\mathcal{B},0)}{\mathrm{d}\mathcal{B}^2} - \frac{\mathrm{d}\widetilde{\mathfrak{L}}(\mathcal{B},0)}{\mathcal{B}\mathrm{d}\mathcal{B}}\right),$$
$$\left.\frac{\partial^2\mathfrak{L}(\mathfrak{F},\mathfrak{G})}{\partial\mathfrak{G}^2}\right|_{\mathfrak{G}=0} = \frac{1}{2\mathfrak{F}}\left.\left(\frac{1}{\mathcal{E}}\frac{\partial\widetilde{\mathfrak{L}}(\mathcal{B},\mathcal{E})}{\partial\mathcal{E}}\right)\right|_{\mathcal{E}=0} + \frac{1}{2\mathfrak{F}}\frac{1}{\mathcal{B}}\frac{\mathrm{d}\widetilde{\mathfrak{L}}(\mathcal{B},0)}{\mathrm{d}\mathcal{B}}. \tag{24}$$

At this step we turn to the special case of magnetic-like background and shall be sticking to it until the end of the present Subsection, keeping the extension of some results to the electric-like case $\mathfrak{F} < 0$ to the next Subsection C.

The dispersion curves $f_a(k_\perp^2)$ near the origin may be found by solving equations (9) in the special frame with the right-hand sides taken as (21) and with eqs. (10) taken into account. This gives the linear functions for photons of modes 2 and 3

$$f_2(k_\perp^2) = k_\perp^2\left(\frac{1-\mathfrak{L}_\mathfrak{F}}{1-\mathfrak{L}_\mathfrak{F}+2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}\right), \tag{25}$$

$$f_3(k_\perp^2) = k_\perp^2\left(1-\frac{2\mathfrak{F}\,\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}{1-\mathfrak{L}_\mathfrak{F}}\right), \tag{26}$$

where we are using the notations $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = \frac{\mathrm{d}^2\mathfrak{L}(\mathfrak{F},0)}{\mathrm{d}\mathfrak{F}^2}$, $\mathfrak{L}_\mathfrak{F} = \frac{\mathrm{d}\mathfrak{L}(\mathfrak{F},0))}{\mathrm{d}\mathfrak{F}}$, $\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \left.\frac{\partial^2\mathfrak{L}(\mathfrak{F},\mathfrak{G})}{\partial\mathfrak{G}^2}\right|_{\mathfrak{G}=0}$. As for mode 1, the dispersion equation in the present approximation has only the trivial solution $k^2 = 0$ that makes the vector potential $\flat_\mu^{(1)}$ corresponding to it purely longitudinal, with no electromagnetic field carried by the mode. This is a nonpropagating mode in the infrared limit (it is also nonpropagating within the one-loop approximation beyond this limit; however, massive-positronium solutions in mode 1 do propagate [32]).

The unitarity condition (16), as applied to mode 2, gives via the second equation in (21)

$$1-\mathfrak{L}_\mathfrak{F}+2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \geq 0. \tag{27}$$

Then, from the behavior of the dispersion curve (25) and the causality (15) it follows that

$$1-\mathfrak{L}_\mathfrak{F} \geq 0 \tag{28}$$

and

$$\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \geq 0. \tag{29}$$

(Remind that for the magnetic-like case under consideration one has $\mathfrak{F} > 0$.)

Analogously, the unitarity condition (16), as applied to mode 3, gives via the third equation in (21) again the result (28). (This inequality also provides the positiveness of the norm of the non-propagating mode 1.) Then from the behavior of the dispersion curve (26) and the causality (15) it follows that

$$1 - \mathfrak{L}_{\mathfrak{F}} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \geq 0 \tag{30}$$

and

$$\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \geq 0. \tag{31}$$

Inequalities eq.(28), eq.(30) together provide that all the three residues of the photon Green function in the complex plane of $k_\perp^2$, the same as in the complex plane of $(k_3^2 - k_0^2)$, eq.(16), are also nonnegative

$$1 - \left.\frac{\partial \varkappa_a(k^2, k_\perp^2, \mathfrak{F})}{\partial k_\perp^2}\right|_{k_0^2 - k_3^2 = f_a(k_\perp^2)} \geq 0, \tag{32}$$

at least in the infrared limit. We do not know whether this statement is prescribed by general principles and therefore might be expected to hold beyond this limit.

Relations (29), (31) indicate that the Lagrangian is a positively (downward) convex function of $\mathfrak{F}$ for any $\mathfrak{F} > 0$ and of $\mathfrak{G}$ in the point $\mathfrak{G} = 0$.

Relations (27), (28), (30) indicate positiveness of various dielectric and magnetic permittivity constants that control electro- and magneto-statics of charges and currents of certain configurations. Eqs. (21) imply that the quantities that are subject to the inequalities (27), (28) and (30) are expressed in terms of different infra-red limits of the polarization operator eigenvalues as

$$\begin{aligned} 1 - \mathfrak{L}_{\mathfrak{F}} &= \lim_{k_\perp^2 \to 0} \left(1 - \frac{\varkappa_2|_{k_0 = k_3 = 0}}{k_\perp^2}\right) \equiv \varepsilon_{\rm tr}(0), \\ 1 - \mathfrak{L}_{\mathfrak{F}} &= \lim_{k_\perp^2 \to 0} \left(1 - \frac{\varkappa_1|_{k_0 = k_3 = 0}}{k_\perp^2}\right) \equiv \left(\mu_{\rm tr}^{\rm w}(0)\right)^{-1}, \\ 1 - \mathfrak{L}_{\mathfrak{F}} &= \lim_{k_3^2 \to 0} \left(1 - \frac{\varkappa_3|_{k_0 = k_\perp = 0}}{k_3^2}\right) \equiv \left(\mu_{\rm long}^{\rm pl}(0)\right)^{-1}, \end{aligned} \tag{33}$$

$$1 - \mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \lim_{k_3^2 \to 0} \left(1 - \frac{\varkappa_2|_{k_0 = k_\perp = 0}}{k_3^2}\right) \equiv \varepsilon_{\rm long}(0), \tag{34}$$

$$1-\mathfrak{L}_{\mathfrak{F}}-2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}=\lim_{k_\perp^2\to 0}\left(1-\frac{\varkappa_3|_{k_0=k_3=0}}{k_\perp^2}\right)\equiv\left(\mu_{\rm tr}^{\rm pl}(0)\right)^{-1}. \tag{35}$$

It is demonstrated in Appendix of Ref. [13] that $\varepsilon_{\rm long}$ and $\varepsilon_{\rm tr}$ are dielectric constants responsible for polarizing the homogeneous electric fields parallel and orthogonal to the external magnetic field, which are produced, respectively, by uniformly charged planes (sufficiently far from them as compared with the formation length of the polarization operator), oriented across the external magnetic field and parallel to it, see eqs.(123) and (125) of [13]. These are determined by the eigenvalue $\varkappa_2$, the virtual photons of the mode 2 being carriers of electrostatic force.

The quantity $\mu_{\rm tr}^{\rm w}(0)$ is the magnetic permittivity constant responsible for attenuation of the magnetic field produced by a constant current concentrated on a line, parallel to the external magnetic field, sufficiently far from the current-carrying line, see Ref. [13] eq.(110) with $\mu(0)$ replaced by $\mu_{\rm tr}^{\rm w}(0)$ in it. The same quantity $\mu_{\rm tr}^{\rm w}(0)$ governs the constant magnetic field of a plane current flowing along the external field. This magnetic permittivity is determined by the mode 1. The other two magnetic permittivities, $\mu_{\rm long}^{\rm pl}(0)$ and $\mu_{\rm tr}^{\rm pl}(0)$ are determined by the mode 3. The permittivity $\mu_{\rm tr}^{\rm pl}(0)$ is responsible for remote attenuation of the magnetic field produced by a constant current, homogeneously concentrated on a plane, parallel to the external magnetic field, and flowing in the direction transverse to it, see Ref. [13] eq.(135). This magnetic field is homogeneous and parallel to the external field. Finally, permittivity $\mu_{\rm long}^{\rm pl}(0)$ is responsible for remote attenuation of the magnetic field produced by a constant straight current, homogeneously concentrated on a plane, transverse to the external magnetic field, see Ref. [13] eq.(138). This field is also homogeneous. Virtual photons of the modes 1 and 3 are carriers of magneto-static force.

By using the wordings "sufficiently far" and "remote" we mean distances from the corresponding sources that essentially exceed a characteristic length of an underlying microscopic theory, wherein the linear response is formed. In a material medium that may be an interatomic distance; in perturbative QED this is the electron Compton length.

Relations (33), (34), (35) mean that the inequalities (27), (28) and (30) signify the positiveness of all the characteristic permittivities of the magnetized vacuum, which was derived above on general basis. Besides, thanks to (33), there exists the equality between one dielectric and two (inverse) magnetic permittivities

$$\varepsilon_{\rm tr}(0)=\left(\mu_{\rm tr}^{\rm w}(0)\right)^{-1}=\left(\mu_{\rm long}^{\rm pl}(0)\right)^{-1}. \tag{36}$$

The first equality here is a direct consequence of the invariance under the Lorentz boost along the magnetic field in the special frame (see eq. (73) in [13] and can be extended to the permittivity functions as defined in [13] by (128) and the right equation (121), $\varepsilon_{\rm tr}(k_\perp^2) = (\mu^{\rm w}_{\rm tr}(k_\perp^2))^{-1}$.

Relations (33) – (35) together with (29), (30) also mean that the longitudinal dielectric constant should be always larger than the transversal one

$$\varepsilon_{\rm long}(0) \geq \varepsilon_{\rm tr}(0), \tag{37}$$

while the magnetic permittivities should satisfy the opposite inequality

$$\mu^{\rm pl}_{\rm tr}(0) \geq \mu^{\rm pl}_{\rm long}(0). \tag{38}$$

C. Electriclike background field

In this subsection we shall see how the inequalities (27)–(31) derived in the previous Subsection are extended to the negative domain of the invariant $\mathfrak{F}$.

Bearing in mind eqs. (11) we may solve again dispersion equations (12) using eqs. (21) to get the photon dispersion curves in the electriclike background field in the infrared approximation. For mode 2 this results in

$$k_0^2 - k_3^2 = k_\perp^2 \left(1 + \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{1 - \mathfrak{L}_\mathfrak{F}}\right), \tag{39}$$

while for mode 3 in

$$k_0^2 - k_3^2 = k_\perp^2 \left(\frac{1 - \mathfrak{L}_\mathfrak{F}}{1 - \mathfrak{L}_\mathfrak{F} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}}\right) \tag{40}$$

(compare this with (25), (26)). The unitarity relation (16) applied to mode 2 leads to the inequality (28). The causality condition (15), when applied to (39) requires that

$$\left(1 + \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{1 - \mathfrak{L}_\mathfrak{F}}\right)^2 \leq \left(1 + \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{1 - \mathfrak{L}_\mathfrak{F}}\right). \tag{41}$$

This implies that the right-hand side of the inequality (41) be positive and thus the both sides can be divided on it. Then the inequality (41) becomes the inequality (27)

$$\left(1 + \frac{2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}}{1 - \mathfrak{L}_\mathfrak{F}}\right) < 1. \tag{42}$$

In view of (28) this means that $2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} < 0$. Once $\mathfrak{F}$ is negative for the electric -like case under consideration now, we come again to the convexity condition (29), now in the domain of negative $\mathfrak{F}$. By applying the same procedure to mode 3 we quite analogously reproduce eqs. (30) and (31).

D. Energy-momentum conditions

Now we proceed with describing general restrictions imposed by the physical requirement that the energy density of elementary excitations of the magnetic-like background (magnetized vacuum) be nonnegative ("weak energy condition" in terms of Ref. [6])

$$t_{00} \geq 0 \tag{43}$$

and that their energy-momentum flux density be non-spacelike ("dominant energy condition" of Ref. [6]))

$$t_{0\nu}^2 \leq 0 \tag{44}$$

in order to compare the results with the conclusions of Subsection B.

We have to define the energy-momentum tensor $t_{\mu\nu}(x)$ of small perturbations of the background field by first defining their Lagrangian. The total effective Lagrangian $L_{\rm tot} = -\mathfrak{F} + \mathfrak{L}$ expanded near the background constant magnetic field contributes into the total action – in view of the definition (2) – the following correction, quadratic in the small perturbation $a_\mu(x)$ above the background:

$$S_{\rm tot}^{\rm sqr} = \frac{1}{2}\int a_\mu(x)\{-\left(\delta_{\mu\nu}\partial_\alpha^2 - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right)\delta(x-y) + \Pi_{\mu\nu}(x,y)\}a_\nu(y){\rm d}^4x{\rm d}^4y. \tag{45}$$

The field intensity of the perturbation will be denoted as $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Using the diagonal form of the polarization operator (3) we get in the momentum representation

$$L_{\rm tot}^{\rm sqr}(k) = \frac{1}{4}f^2 + \frac{1}{4}\left(-\frac{\varkappa_1}{k^2}f^2 + \frac{\varkappa_1-\varkappa_2}{2k\widetilde{F}^2k}((f\widetilde{F}))^2 + \frac{\varkappa_1-\varkappa_3}{2kF^2k}((fF))^2\right). \tag{46}$$

Here the notations are used: $(fF)_{\mu\nu} = f_{\mu\alpha}F_{\alpha\nu} = (Ff)_{\nu\mu}$, $(fF) = (fF)_{\mu\mu} = (Ff)$, $f^2_{\ \mu\nu} = f_{\mu\alpha}f_{\alpha\nu}$, $f^2 = f^2_{\ \mu\mu} = -(f_{\mu\nu})^2$, and we have exploited the relations $f^2 = -2a_\mu(k^2\delta_{\mu\nu} - k_\mu k_\nu)a_\nu$, $(fF) = 2(aFk)$. This Lagrangian is nonlocal, since it depends on momenta in a complicated way, in other words, it depends highly nonlinearly on the derivatives with

respect to coordinates. It becomes local if we restrict ourselves to the infrared limit by substituting eqs.(21) into it. Then the quadratic Lagrangian acquires the very compact form

$$L_{\rm tot}^{\rm sqr} = \frac{1}{4}f^2(1-\mathfrak{L}_{\mathfrak{F}}) + \frac{1}{8}\left(\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}((f\widetilde{F}))^2 + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}((fF))^2\right). \tag{47}$$

This Lagrangian, quadratic in the field $f_{\mu\nu}(x)$, does not contain its derivatives, $F_{\mu\nu}, \widetilde{F}_{\mu\nu}, \mathfrak{L}_{\mathfrak{F}}, \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}$ being constants depending upon the background field alone. It governs small-amplitude low-frequency and low-momentum perturbations of the magnetized vacuum, free of or created by small sources. It might be obtained also directly by calculating the second derivative (2) of the Lagrangian defined on constant fields [14].

Once the background is translation-invariant, there is a conserved energy-momentum tensor $t_{\mu\nu}(x)$ of the field $f_{\mu\nu}$ provided by the Noether theorem by considering variations of this field. Applying the standard definition of the energy-momentum tensor to the field of small perturbation a_μ and to its Lagrangian (47) we get

$$\begin{aligned} t_{\mu\nu}(x) = -\frac{\partial L_{\rm tot}^{\rm sqr}}{\partial(\partial a_\alpha/\partial x_\nu)}\frac{\partial a_\alpha}{\partial x_\mu} + \delta_{\mu\nu}L_{\rm tot}^{\rm sqr} = \\ = -\frac{\partial a_\alpha}{\partial x_\mu}\left(f_{\alpha\nu}(1-\mathfrak{L}_{\mathfrak{F}}) + \frac{1}{2}(f\widetilde{F})\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\widetilde{F}_{\alpha\nu} + \frac{1}{2}(fF)\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}F_{\alpha\nu}\right) + \delta_{\mu\nu}L_{\rm tot}^{\rm sqr}. \end{aligned} \tag{48}$$

The Maxwell equations for small sourceless perturbations of the magnetized vacuum are

$$\frac{\delta L_{\rm tot}^{\rm sqr}}{\delta a_\alpha} = \frac{\partial}{\partial x_\nu}\frac{\partial L_{\rm tot}^{\rm sqr}}{\partial(\partial a_\alpha/\partial x_\nu)} = \frac{-\partial}{\partial x_\nu}\left(f_{\alpha\nu}(1-\mathfrak{L}_{\mathfrak{F}}) + \frac{1}{2}(f\widetilde{F})\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\widetilde{F}_{\alpha\nu} + \frac{1}{2}(fF)\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}F_{\alpha\nu}\right) = 0. \tag{49}$$

We are going to use the standard indeterminacy in the definition of the energy-momentum tensor to let it depend only on the field strength $f_{\mu\nu}$, and not on its potential. To this end we add the quantity (the designation $\doteq$ below means "equal up to full derivative")

$$\begin{aligned} \frac{\partial L_{\rm tot}^{\rm sqr}}{\partial(\partial a_\alpha/\partial x_\nu)}\frac{\partial a_\mu}{\partial x_\alpha} \doteq -a_\mu\frac{\partial}{\partial x_\alpha}\frac{\partial L_{\rm tot}^{\rm sqr}}{\partial(\partial a_\alpha/\partial x_\nu)} = \\ = a_\mu\frac{\partial}{\partial x_\alpha}\{f_{\alpha\nu}(1-\mathfrak{L}_{\mathfrak{F}}) + \frac{1}{2}(f\widetilde{F})\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\widetilde{F}_{\alpha\nu} + \frac{1}{2}(fF)\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}F_{\alpha\nu}\} \end{aligned} \tag{50}$$

to (48), that disappears due to the Maxwell equations (49), taking into account the antisymmetricity of the expression inside the braces. Hence the energy-momentum tensor may be equivalently written as

$$\begin{aligned} t_{\mu\nu}(x) = -f^2{}_{\mu\nu}(1-\mathfrak{L}_{\mathfrak{F}}) - \frac{1}{2}(f\widetilde{F})\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(f\widetilde{F})_{\mu\nu} - \frac{1}{2}(fF)\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(fF)_{\mu\nu} + \\ + \frac{\delta_{\mu\nu}}{4}\left(f^2(1-\mathfrak{L}_{\mathfrak{F}}) + \frac{1}{2}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}((f\widetilde{F}))^2 + \frac{1}{2}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}((fF))^2\right). \end{aligned} \tag{51}$$

This tensor is traceless, $t_{\mu\mu} = 0$. It obeys the continuity equation with respect to the *second* index

$$\frac{\partial t_{\mu\nu}}{\partial x_\nu} = 0 \tag{52}$$

owing to the Maxwell equations (49). Hence, the 4-momentum vector obtained by integrating $t_{0\mu}$ over the spatial volume d^3x conserves in time.

Let us take (51), first, on the monochromatic – with 4-momentum k_μ – real solution of the Maxwell equations (49) that belongs to the eigen-mode 3: $f^{(3)}_{\mu\nu} = k_\mu \flat^{(3)}_\nu - k_\nu \flat^{(3)}_\mu$. One has $(f^{(3)}F)_{\mu\nu} = \flat^{(3)}_\mu \flat^{(3)}_\nu - k_\mu(F^2k)_\nu$, $(f^{(3)}F) = -2(kF^2k)$, $(f^{(3)})^2_{\mu\nu} = -k^2\flat^{(3)}_\mu \flat^{(3)}_\nu + k_\mu k_\nu(kF^2k)$, $(f^{(3)})^2 = 2k^2(kF^2k)$, $(f^{(3)}\widetilde{F}) = 0$. With the substitution $f_{\mu\nu} = f^{(3)}_{\mu\nu}$ the Maxwell equation (49) is satisfied, when

$$\flat^{(3)}_\alpha \{k^2(1-\mathfrak{L}_\mathfrak{F}) + (kF^2k)\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}\} = 0, \tag{53}$$

i.e., naturally, on the dispersion curve (26) for mode 3. It is seen that the Lagrangian (47) disappears on the mass shell of mode 3, $L^{\mathrm{sqr}(3)}_{\mathrm{tot}} = 0$. Then, the reduction of the energy momentum tensor (51) onto this mode, $t^{(3)}_{\mu\nu}(x)$, should be written with its $\delta_{\mu\nu}$ part dropped:

$$t^{(3)}_{\mu\nu}(x) = (1-\mathfrak{L}_\mathfrak{F})(k^2\flat^{(3)}_\mu \flat^{(3)}_\nu - k_\mu k_\nu(kF^2k)) + (kF^2k)\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(\flat^{(3)}_\mu \flat^{(3)}_\nu - k_\mu(F^2k)_\nu). \tag{54}$$

Then, after omitting the common factor $-(kF^2k)$ equal to $2\mathfrak{F}k^2_\perp > 0$ in a magnetic field, and to $2\mathfrak{F}(k_0^2 - k_3^2) > 0$ in an electric field, and using the mass shell equation once again, we get

$$t^{(3)}_{\mu\nu}(x) = (1-\mathfrak{L}_\mathfrak{F})k_\mu k_\nu + k_\mu \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(F^2k)_\nu. \tag{55}$$

Although we referred to the magnetic-like background above in this Subsection, all the equations written in it up to now remain, as a matter of fact, valid also for the electric-like case. In the rest of this Subsection we actually specialize to the magnetized vacuum, although the conclusions may be readily extended to cover the electrified vacuum, as well. When $\mathfrak{F} > 0$, in the special frame $(F^2k)_{0,3} = 0$, $(F^2k)_{1,2} = -2\mathfrak{F}k_{1,2}$. It is convenient to write the energy-momentum density vector in components (counted as 0,1,2,3 downwards)

$$t^{(3)}_{0\nu} = k_0 \begin{pmatrix} k_0(1-\mathfrak{L}_\mathfrak{F}) \\ k_1(1-\mathfrak{L}_\mathfrak{F} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}) \\ k_2(1-\mathfrak{L}_\mathfrak{F} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}) \\ k_3(1-\mathfrak{L}_\mathfrak{F}) \end{pmatrix}_\nu . \tag{56}$$

The spacial part of this vector density is parallel to the group velocity $\mathbf{v}_{\rm gr}^{(3)} = ({\rm d}k_0/{\rm d}\mathbf{k})$ calculated on the mode-3-mass-shell as defined by the dispersion law (13), (26)

$$t_{0i}^{(3)} = (v_{\rm gr}^{(3)})_i k_0(1-\mathfrak{L}). \tag{57}$$

The positive definiteness of the energy density (43) results again in the requirement that the inequality (28) be satisfied. The causality in the form of the dominant energy condition (44) makes us expect that vector (56) should be non-spacelike. Now, from (56) with the use of the dispersion law (26) this condition becomes

$$\begin{aligned} t_{0,\mu}^{(3)2} &= k_0^2\{(k_3^2-k_0^2)(1-\mathfrak{L}_{\mathfrak{F}})^2 + k_\perp^2(1-\mathfrak{L}_{\mathfrak{F}}-2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}})^2\} = \\ &= -2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}k_0^2k_\perp^2(1-\mathfrak{L}_{\mathfrak{F}}-2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}) \leq 0. \end{aligned} \tag{58}$$

Owing to relation (57), this is exactly equivalent to the requirement (14) that the group velocity of mode-3 photons should not exceed the speed of light in the vacuum.

The same operations, performed over the energy-momentum tensor (51) taken on mode 2, result (after omitting the positive factor -$k\widetilde{F}^2k$) in an expression for the energy-momentum tensor $t_{\mu\nu}^{(2)}$ that is obtained from (55) by the duality transformation $F \to \widetilde{F}$, $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \to \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}$. When $\mathfrak{F} > 0$, in the special frame $(F^2k)_{1,2} = 0$, $(F^2k)_{0,3} = 2\mathfrak{F}k_{0,3}$, so

$$t_{0\nu}^{(2)} = k_0 \begin{pmatrix} k_0(1-\mathfrak{L}_{\mathfrak{F}}+2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}) \\ k_1(1-\mathfrak{L}_{\mathfrak{F}}) \\ k_2(1-\mathfrak{L}_{\mathfrak{F}} \\ k_3(1-\mathfrak{L}_{\mathfrak{F}}+2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}) \end{pmatrix}_\nu . \tag{59}$$

The positivity of the energy density $t_{00}^{(2)}$ leads to the inequality (27). The group velocity of mode 2 is again parallel to the momentum density 3-vector

$$t_{0i}^{(2)} = (v_{\rm gr}^{(2)})_i k_0(1-\mathfrak{L}_{\mathfrak{F}}+2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}). \tag{60}$$

The causality in the form of the dominant energy condition (44) leads from (59) with the use of the dispersion law (25) to

$$t_{0\mu}^{(2)2} = -2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}k_0^2(k_0^2-k_3^2)(1-\mathfrak{L}_{\mathfrak{F}}+2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}) \leq 0. \tag{61}$$

Owing to relation (60), this is exactly equivalent to the requirement (14) that the group velocity of mode-2 photons should not exceed the speed of light in the vacuum. Bearing in mind that eq. (27) is already established, eq. (29) follows from (61).

To resume, we were able to reproduce in this Subsection the requirements (27)–(29), but the remaining requirements (30) and (31) do not follow from (58), although the latter does not contradict them. Since, as it was explained, the form of the causality conditions (44) used in this Subsection is equivalent to the group velocity restriction (14), we think that our analysis has indicated that the energy-density nonnegativity (43) condition is somewhat weaker than the unitarity condition in the form (16).

The fulfillment of (58), (61) is guaranteed by the inequalities (27), (29)– (30) established in Subsection B. However, the inverse statement would be wrong: the inequalities (58), (61), derived in the present Subsection do not yet lead to (27), (29)– (30). This may indicate that pair of conditions (16) (unitarity as the positivity of the residue) and (14) (causality as the boundedness of the group velocity), used to derive the limitations (27) – (30) of Subsection B, are together more restrictive than the two principles (43) (energy positiveness) and (44) (causality as non-spacelikeness of the energy-momentum density), although the latter provide the fact that when solving the Cauchy problem initial data have no influence on what occurs outside their light cone. (This is proved in [6] within General Relativity context.)

III. TESTING CERTAIN LAGRANGIANS

A. Euler-Heisenberg effective Lagrangian

In the one-loop approximation of QED the quantities involved can be calculated either using the Euler-Heisenberg effective Lagrangian $\mathfrak{L} = \mathfrak{L}^{(1)}$ [15], when the infrared limit is concerned, or, alternatively, the one-loop polarization operator calculated in [4] for off-shell photons – within and beyond this limit. In the infrared limit the photon-momentum-independent coefficients in (21) within one loop are the following functions of the dimensionless magnetic field $b = eB/m^2$, where e and m are the electron charge and mass:

$$\mathfrak{L}^{(1)}_{\mathfrak{F}} = \frac{\alpha}{2\pi}\int_0^\infty \frac{\mathrm{d}t}{t}\exp\left(-\frac{t}{b}\right)\left(\frac{-\coth\, t}{t} + \frac{1}{\sinh^2 t} + \frac{2}{3}\right), \tag{62}$$

$$2\mathfrak{F}\mathfrak{L}^{(1)}_{\mathfrak{G}\mathfrak{G}} = \frac{\alpha}{3\pi}\int_0^\infty \frac{\mathrm{d}t}{t}\exp\left(-\frac{t}{b}\right)\left(\frac{-3\coth\, t}{2t} + \frac{3}{2\sinh^2 t} + t\coth t\right), \tag{63}$$

$$2\mathfrak{F}\mathfrak{L}^{(1)}_{\mathfrak{F}\mathfrak{F}} = \frac{\alpha}{2\pi}\int_0^\infty \frac{\mathrm{d}t}{t}\exp\left(-\frac{t}{b}\right)\left(\frac{\coth\, t}{t} - \frac{2t\coth t}{\sinh^2 t} + \frac{1}{\sinh^2 t}\right). \tag{64}$$

Here $\alpha = e^2/4\pi = 1/137$ is the fine-structure constant. (We refer to the Heaviside-Lorentz system of units with $c = \hbar = 1$). Eq. (62) turns to zero as $\mathfrak{F} \sim b^2$, since the divergent linear in $\mathfrak{F}$ part of the one-loop diagram was absorbed in the course of renormalization into $\mathfrak{L}_{\rm cl}$. It can be verified that the general relations (27)–(31) ordained by unitarity (16) and causality (15) to the infrared limit are obeyed by the one-loop approximation within the vast range of the magnetic field values. (We are not considering in the present context the electric-like case, since the (one-loop) Heiseberg-Euler Lagrangian suffers the known instability under spontaneous production of electron-positron pairs.) However, due to the known lack of asymptotic freedom in QED [16], some of the general relations are violated for exponentially strong fields of Planck scale. One can establish the asymptotic behavior of (62) - (64) in the limit $b = eB/m^2 \to \infty$

$$\mathfrak{L}_{\mathfrak{F}}^{(1)} \simeq \frac{\alpha}{3\pi}(\ln b - 1.79), \qquad 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}^{(1)} \simeq \frac{\alpha}{3\pi}(b - 1.90), \qquad 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}^{(1)} \simeq \frac{\alpha}{3\pi}. \tag{65}$$

One can see then that the convexity properties (29), (31) and hence the inequalities (37), (38) are left intact under arbitrarily strong magnetic field within one loop. So is the inequality (27), thanks to the linearly growing [17] term in $\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}^{(1)}$. On the contrary, eq.(30) is violated for $b > b_1^{\rm cr} = \exp\{0.79 + 3\pi/\alpha\}$, and eq. (28) for $b > b_2^{\rm cr} = \exp\{1.79 + 3\pi/\alpha\} > b_1^{\rm cr}$.

Let us inspect consequences of these violations. First note that the inequality (15) requires that $f_a(k_\perp^2) \geq 0$, hence no branch of any dispersion curve may get into the region $k_0^2 - k_3^2 < 0$. If it might, the photon energy k_0 would have an imaginary part within the momentum interval $0 < k_3^2 < -f_a(k_\perp^2)$, corresponding to the vacuum excitation exponentially growing in time. This sort of ghost would signal the instability of the magnetized vacuum. Inequality (15) further requires that

$$\frac{{\rm d} f_a^{\frac{1}{2}}(k_\perp^2)}{{\rm d} k_\perp} \leq 1, \qquad \text{or} \qquad f_a^{\frac{1}{2}}(k_\perp^2) \leq const + k_\perp. \tag{66}$$

All the dispersion curves (25), (26) in the infrared approximation we are dealing with correspond to zero-mass vacuum excitations $k_0|_{k_3=k_\perp=0} = 0$ – photons, since $f(0) = 0$. Therefore $const = 0$.

Consider, first, mode 2. We mentioned that relation (27), which is the positive-norm condition for this mode, is fulfilled for any large b. When $b < b_2^{\rm cr}$, also the dispersion curve goes outside the light cone, $\sqrt{k_0^2 - k_3^3} \leq k_\perp$, as it is prescribed by eq. (66) with $const = 0$.

However, the bracket in (25) becomes negative for $b > b_2^{\rm cr}$, and mode 2 becomes a complex energy ghost.

Now comes mode 3. The positive norm condition for it, (relation (28)), is fulfilled, when $b < b_2^{\rm cr}$. However, within the range $b_1^{\rm cr} < b < b_2^{\rm cr}$ the bracket in (26) is negative, and mode 3 is a complex energy ghost. For $b > b_2^{\rm cr}$ the dispersion curve (26) for mode-3 photon gets inside the light cone, $\sqrt{k_0^2 - k_3^3} \geq k_\perp$, in contradiction with eq. (66) and thus becomes a super-luminal excitation, tachyon, with real energy and negative norm. Note, that these superluminal excitations, peculiar to mode 3, can hardly appear in reality, since the background field becomes unstable before it can reach, when growing, the necessary critical value $b = b_2^{\rm cr}$. An instability of the magnetized vacuum with respect to production of a constant field is associated with the imaginary energy at zero momentum. The elementary excitation with this property appears in mode 3 at a smaller threshold value, $b_3^{\rm cr}$, than in mode 2, $b_2^{\rm cr}$. The instability associated with mode-2 ghosts may lead to gaining the constant field with $\mathfrak{G} \neq 0$, since the (pseudo)vector-potential $\flat_\mu^{(2)}$ (4) carries an electric field component, parallel to the background magnetic field, whereas in $\flat_\mu^{(3)}$ this component is perpendicular to $\mathbf{B}$.

The borders of stability of the magnetic field found here by analyzing the one-loop approximation are characterized by the large exponential $\exp\{1/\alpha\}$. It is much larger than the border found earlier [18] as the value where the mass defect of the bound electron-positron pair completely compensates the $2m$ energy gap between the electron and positron, which is of the order of $\exp\{1/\sqrt{\alpha}\}$.

B. Born-Infeld Lagrangian

The situation is quite different for the Born-Infeld electrodynamics with its Lagrangian

$$L_{\rm tot} = L^{\rm BI} = a^2 \left(1 - \sqrt{1 + \frac{2\mathfrak{F}}{a^2} - \frac{\mathfrak{G}^2}{a^4}}\right) \qquad (67)$$

viewed upon as final, not subject to further quantization. Here a is an arbitrarily large parameter with the dimensionality of mass squared. The correspondence principle (1) is respected by eq. (67). It does not contain field derivatives, hence all the infra-red limits encountered in this paper should be understood as exact values, for instance, going to the limit is unnecessary in (33), (34), (35). The Lagrangian (67) was derived long ago [19]

basing on very general geometrical principles of reparametrization-invariance, and besides it attracted much attention in recent decades thanks to the fact that it appears responsible for the electromagnetic sector of a string theory [20] and thus is expected not to suffer from the lack of asymptotic freedom. For this reason our statement to follow that all the fundamental requirements established in Section 2 are obeyed in the Born-Infeld electrodynamics (67) is instructive. We assume again that there is the constant and homogeneous magnetic-like external background and set $\mathfrak{G}=0$ after differentiation. Then, we get from (67)

$$1-\mathfrak{L}^{\rm BI}_{\mathfrak{F}}=\left(1+\frac{2\mathfrak{F}}{a^2}\right)^{-\frac{1}{2}}\geq 0,\quad \mathfrak{L}^{\rm BI}_{\mathfrak{F}\mathfrak{F}}=a^{-2}\left(1+\frac{2\mathfrak{F}}{a^2}\right)^{-\frac{3}{2}}\geq 0,\quad \mathfrak{L}^{\rm BI}_{\mathfrak{G}\mathfrak{G}}=a^{-2}\left(1+\frac{2\mathfrak{F}}{a^2}\right)^{-\frac{1}{2}}\geq 0,$$
$$1-\mathfrak{L}^{\rm BI}_{\mathfrak{F}}+2\mathfrak{F}\mathfrak{L}^{\rm BI}_{\mathfrak{G}\mathfrak{G}}=\left(1+\frac{2\mathfrak{F}}{a^2}\right)^{\frac{1}{2}}\geq 0,\qquad 1-\mathfrak{L}^{\rm BI}_{\mathfrak{F}}-2\mathfrak{F}\mathfrak{L}^{\rm BI}_{\mathfrak{F}\mathfrak{F}}=\left(1+\frac{2\mathfrak{F}}{a^2}\right)^{-\frac{3}{2}}\geq 0 \tag{68}$$

where $\mathfrak{L}^{\rm BI}=L^{\rm BI}+2\mathfrak{F}$. Thus, relations (27)–(31) are all satisfied, hence there are neither ghosts, nor tachyons. The mode 1 remains nonpropagating. As for modes 2 and 3, their dispersion curves coincide, since $f_2(k_\perp^2)=f_3(k_\perp^2)$ in (25), (26) due eqs. (68). This reflects the known absence of birefringence in the Born-Infeld electrodynamics [21]. Still, beyond the mass shell one has $\varkappa_2\neq\varkappa_3$, consequently the corresponding permeabilities (33), (34), (35) are different. The same as in the one-loop QED, in the limit of large external field there is a linearly growing contribution in $\varkappa_2$, so mode 2 dominates, the dielectric permeability (34) behaving like the middle equation in (65)

$$\varepsilon^{\rm BI}_{\rm long}(0)\simeq 2\mathfrak{F}\mathfrak{L}^{\rm BI}_{\mathfrak{G}\mathfrak{G}}\simeq\frac{B}{a} \tag{69}$$

with the identification $a=(3\pi/\alpha)B_0$, where $B_0=m^2/e=4.4\times 10^{14}$ Gauss is the characteristic field strength in QED. As a matter of fact, however, it is believed that a should be of the Planck scale $a\simeq m_{\rm Pl}^2/e=5.8\cdot 10^{44}B_0$.

If we include the electric-like case we shall see that eqs. (68) are all fulfilled within the interval $-(a^2/2)<\mathfrak{F}<\infty$, at the border of which the Lagrangian (67) becomes imaginary (recall that $\mathfrak{G}=0$.)

C. Lagrangians giving rise to spontaneous magnetic field

In this Subsection we consider, as counterexamples, two effective Lagrangians that lead to nonzero magnetic field as the minimum energy point and are thus conventionally interpreted

as spontaneously producing a constant homogeneous magnetic field $B_{\rm sp}$. In both of these cases below, one of which relating to a nonAbelian gauge theory, the fundamental properties of the Lagrangian established in Section IIB are violated in and around the point $B = B_{\rm sp}$.

1. Batalin-Matinian-Savvidy Lagrangian

These authors calculated [7] – with the one-loop accuracy and using Schwinger's proper-time method – the effective Lagrangian in the Yang-Mills theory as a function of two time- and space-independent field invariants.

The intensity tensor $G^a_{\mu\nu} = \partial A^a_\mu - \partial A^a_\nu - g\epsilon^{abc}A^b_\mu A^c_\nu$ is subject to the sourceless equation

$$\nabla^{ab}_\nu G^b_{\nu\mu} = 0 \tag{70}$$

with the standard covariant derivative $\nabla^{ab}_\mu = \delta^{ab}\partial_\mu + gA^{ab}_\mu$, $A_{ab} = \epsilon^{acb}A^c_\mu$. Here the superscript a is responsible for the isotopic degree of freedom, the subscript $\mu = (i, 0)$ runs the space-time components, g is the coupling constant, and ϵ^{abc} are the structural constants of SU(2). The simplest solution of the equation (70) is the covariant constant field that satisfies the equation

$$\nabla^{ab}_\rho G^b_{\nu\mu} = 0. \tag{71}$$

It follows from (71) that the intensity tensor factorizes as $G^a_{\mu\nu} = F_{\mu\nu}n^a$, i.e. it is directed in the isotopic space along a permanent direction of the constant (chosen as unit) isotopic vector n^a, $F_{\mu\nu}$ being a constant tensor, carrying the "chromomagnetic" and "chromoelectric" background fields. In a special gauge the vector potential may be chosen as $A^a_\mu = A_\mu n^a = -(1/2)F_{\mu\nu}x_\nu n^a$. It is seen that the present case is mostly close to quantum electrodynamics, the calculations can be made in a gauge-independent way and the result for the effective Lagrangian depends on the background Abelian field via the field invariants $\mathfrak{F}$ and $\mathfrak{G}$ defined in terms of the tensor $F_{\mu\nu}$ in the same way as in QED.

The polarization operator responsible for propagation of small nonAbelian fields (gluons) against the background considered is, generally, defined by an equation similar to (2)

$$\Pi^{ab}_{\mu\tau}(x,y) = \left.\frac{\delta^2\Gamma}{\delta A^a_\mu(x)\delta A^b_\tau(y)}\right|_{\mathfrak{G}=0,\ \mathfrak{F}=\text{const},\ A^a_\mu=A_\mu n^a}. \tag{72}$$

Then the polarization operator (2) is the projection of (72) to the only isotopic direction

$$\Pi_{\mu\tau}(x,y) = n^a n^b \Pi^{ab}_{\mu\tau}(x,y). \tag{73}$$

This quantity governs the propagation of small perturbations of the background field polarized in the isotopic space parallel to that field (call them chromophotons). The polarization operator (73) possesses all the properties exploited in Section II, hence it makes sense the inspect whether the Batalin-Matinian-Savvidy Lagrangian obeys the properties (27)–(31) relating to propagation of long-wave low frequency chromophotons.

The total Lagrangian is again $L = -\mathfrak{F} + \mathfrak{L}$, where $-\mathfrak{F}$ is the tree Lagrangian on the covariantly constant fields under consideration. After renormalization the one-loop result of Ref. [7] for the real part of the effective Lagrangian $\mathfrak{L}$ can be represented as

$$\mathfrak{L}(\mathfrak{F}, \mathfrak{G}^2) = \widetilde{\mathfrak{L}}(\mathcal{B}, \mathcal{E}) = \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left\{\frac{g^2\mathcal{B}\mathcal{E}}{\sinh(g\mathcal{B}s)\sin(g\mathcal{E}s)} - \frac{1}{s^2} + \frac{g^2(\mathcal{B}^2 - \mathcal{E}^2)}{6}\right\}\mathrm{e}^{-\mu^2 s} + \\ + \frac{1}{4\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s} g^2\left\{\mathcal{E}\mathcal{B}\left[\frac{\sin(g\mathcal{B}s)}{\sinh(g\mathcal{E}s)} - \frac{\sin(g\mathcal{E}s)}{\sinh(g\mathcal{B}s)}\right] + \mathcal{E}^2 - \mathcal{B}^2\right\}\mathrm{e}^{-\mu^2 s}, \tag{74}$$

where the invariant combinations $\mathcal{B}$ and $\mathcal{E}$ are defined by (23) and coincide with the chromo-magnetic and chromo-electric fields in a special Lorentz frame, respectively. The normalization condition, obeyed by (74), contrary to (1), was imposed in a nonzero point

$$\left.\frac{\mathrm{d}\mathfrak{L}(\mathfrak{F}, 0)}{\delta\mathfrak{F}}\right|_{\sqrt{2\mathfrak{F}}=\mu^2} \equiv \frac{1}{\mathcal{B}}\left.\frac{\mathrm{d}\widetilde{\mathfrak{L}}(\mathcal{B}, 0)}{\mathrm{d}\mathcal{B}}\right|_{\mathcal{B}=\mu^2} = 0. \tag{75}$$

The equality here is the first line of (24). The integral in (74) is convergent in the ultraviolet ($s \simeq 0$) and the infrared ($s \simeq \infty$) regions of the proper-time integration variable s.

When $\mathfrak{G} = 0$ and $\mathfrak{F} > 0$, one has $\mathcal{E} = 0$ and $\mathcal{B} = \sqrt{2\mathfrak{F}}$.

$$\mathfrak{L}(\mathfrak{F}, 0) = \widetilde{\mathfrak{L}}(\mathcal{B}, 0) = \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}\left\{\frac{g\mathcal{B}}{s\sinh(g\mathcal{B}s)} - \frac{1}{s^2} + \frac{g^2\mathcal{B}^2}{6}\right\}\mathrm{e}^{-\mu^2 s} + \\ + \frac{1}{4\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s} g^2\left\{\mathcal{B}\frac{\sin(g\mathcal{B}s)}{gs} - \mathcal{B}^2\right\}\mathrm{e}^{-\mu^2 s}, \tag{76}$$

The asymptotic behavior of (74) and of (76) at $\mathfrak{F} \to \infty$ are the same as at $\mu^2 \to 0$, since (74) is a function of the ratio $\mu^2/\mathcal{B}$. Eq. (76) behaves as

$$\mathfrak{L}(\mathfrak{F}, 0) \asymp -\frac{11}{48\pi^2} g^2 \mathfrak{F} \ln\left(\frac{2g^2\mathfrak{F}}{\mu^4}\right). \tag{77}$$

Correspondingly, in the leading order

$$\mathfrak{L}_{\mathfrak{F}} = -\frac{11}{48\pi^2} g^2 \ln\left(\frac{2g^2\mathfrak{F}}{\mu^4}\right), \qquad 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = -\frac{11g^2}{24\pi^2}. \tag{78}$$

It follows from (74) with the use of (24) that

$$2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \frac{1}{\mathcal{E}}\left(\frac{\partial\widetilde{\mathfrak{L}}(\mathcal{B},\mathcal{E})}{\partial\mathcal{E}}\right)\Bigg|_{\mathcal{E}=0} + \frac{1}{\mathcal{B}}\frac{\mathrm{d}\widetilde{\mathfrak{L}}(\mathcal{B},0)}{\mathrm{d}\mathcal{B}} =$$
$$= \frac{g^2}{4\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t}\left\{\frac{-t\sin t}{3} + \frac{\sinh t - t\cosh t}{2t\sinh^2 t} + \right. \tag{79}$$
$$\left. + \frac{\sin t}{t} + \cos t - \frac{11}{6}\frac{t}{\sinh t}\right\}\exp\left(-\frac{\mu^2}{g\mathcal{B}}t\right), \tag{80}$$

where $t = g\mathcal{B}s$. The integral of the first term in the bracket is readily calculated to be equal to -1 in the limit $(\mu^2/g\mathcal{B}) = 0$, whereas the rest of it converges – even without the infrared regularization – to a constant calculated numerically. The convergence of (79) in the limit of infinite magnetic field, unlike the QED expression (63), is the formal reason why the linearly growing contribution to the dielectric permeability of the magnetized vacuum, found responsible for the formation of a string-like Coulomb potential in QED [22], is absent from chromomagnetized vacuum. Finally, in the above limit, we get

$$2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = -\frac{g^2}{4\pi^2}\left(\frac{1}{3} + 1.5...\right) = -\frac{11g^2}{24\pi^2}. \tag{81}$$

Contrary to (65), the contribution, linear in the magnetic field, is not present here.

We see from (78), (81) that the general conditions (28), (27) and (30), derived in Sec. II for the dielectric and magnetic permeabilities, are obeyed, while the convexity properties (29) and (31) are not. So, the chromomagnetized vacuum is free, within the one-loop approximation, of superluminal excitations and ghosts, characteristic of the Euler-Hiesenberg approximation in QED, as described in Subsection A above. On the contrary, the wrong convexity $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} < 0$ results in the fact that the effective potential $V_{\text{eff}} = -\mathfrak{L}$ has its minimum at a nonvanishing value of the magnetic field [23]. Bearing in mind that any constant magnetic field satisfies exact equation of motion without sources due to gauge invariance, it is concluded that the nonzero magnetic field is produced spontaneously. (As distinct to the scalar Higgs case, the equation for potential minimum is not an equation of motion for the gauge field.) However, the shift to the minimum point does not result in improving the wrong convexity sign. The matter is that there is an instability of the magnetic field reflected in appearance of imaginary part of the effective Lagrangian (already for magnetic-like case under consideration) due to contribution of unstable gluon mode in a magnetic field [24] into the spectral decomposition of the effective action. (The presence of the imaginary part

not seen in [7] may be reproduced [25], [9] also in calculations following the Schwinger's proper time technique). This instability is known to be resolved by going out of the sector of covariantly-constant fields.

2. *Kawati-Kokado Lagrangian*

The Lagrangian of the named authors [8] is remarkable in that it proclaims spontaneous production of the magnetic field as large as $10^{45} - 10^{47}$ G in the course of inflation. The model includes interaction between an electromagnetic and a complex massless scalar fields considered in de Sitter space-time. When there is no direct coupling between the scalar field and the de Sitter metric field, the Lagrangian, calculated as a function of a constant magnetic field, which satisfies sourceless equations of motion, is

$$L = -\frac{1}{2}B^2 - \frac{e^2B^2}{192\pi^2}\left(\ln\frac{e^2B^2}{\kappa^2} + \alpha\right) + \frac{H^2\ln 2}{8\pi^2}|eB|, \tag{82}$$

where H is the Hubble constant incorporated in the de Sitter metric, κ is a parameter taken to adjust the dimension, and α is a certain numerical parameter. The convexity of the Lagrangian (82) with respect to the variable $\mathfrak{F} = -B^2/2$ is upward in the region $\mathfrak{F} > 0$, in other words condition (31) is violated throughout the magneticlike domain of $\mathfrak{F}$. As a consequence, the effective potential, which is the Lagrangian taken with the opposite sign, has a minimum at $B = B_{\rm sp}$ with

$$B_{\rm sp} = \frac{eH^2}{8\pi^2}. \tag{83}$$

(The small quantity $(e^2/192\ \pi^2)$ was neglected.) The value of the spontaneous magnetic field listed above is obtained in [8] taking the typical values for the Hubble constant, $H \sim 10^{15} - 10^{17}$Gev, in (83). Its existence is completely due to the violation of the general principles, reflected in eq. (31). Note that, as distinct from the Higgs mechanism, the wrong convexity of the Lagrangian is not improved after the shift to the value (83). The other general requirement, eq. (28), is violated for $B < B_{\rm sp}$. Unlike the QED case of Subsection A, this violation occurs at small values of the magnetic field. (We cannot check conditions (27) and (29), since calculations with the second field-invariant $\mathfrak{G}$ kept different from zero are not available.)

D. Yang-mills field with external source

The one-loop effective Lagrangian as a function of the background Yang-Mills (gluon) field that has a nonvanishing classical source J^a_μ was calculated in [9], [10] within a special quantization procedure needed to substitute for the gauge invariance violated by that source. In this approach the vanishing of the covariant derivative $\nabla^{ac}_k J^c_k(t)$, required by the gauge invariance, is achieved by treating this derivative as the secondary constraint. Correspondingly, under quantization, the functional delta-function $\delta(\nabla^{ac}_k J^c_k(x))$ appears in the functional integral over the gluon field to restrict, in the course of integration, their values involved in this covariant derivative.

Let there be a constant background (classical) SU(2) Yang-Mills potential that in a special Lorentz frame and in a special gauge has the form

$$A^a_i = (A^2/3)^{1/2}\delta^a_i, \qquad A^a_0 = 0, \tag{84}$$

where δ^a_i is the Kronecker symbol and $A^2 = A^a_\mu A^a_\mu$. Here the superscript a is responsible for the isotopic degree of freedom, while the subscript $\mu = (i,0)$ marks the space-time components. The field intensity tensor of the constant potential (84) is $G^a_{\mu\nu} = g\epsilon^{abc}A^b_\mu A^c_\nu$, where g is the selfcoupling constant, and ϵ^{abc} are the SU(2) fully antisymmetric unit tensor. The Yang-Mills equation is

$$\nabla^{ab}_\nu G^b_{\nu\mu} = -\frac{2}{3}g^2A^2A^a_\mu, \tag{85}$$

with the standard covariant derivative $\nabla^{ab}_\mu = \delta^{ab}\partial_\mu + gA^{ab}_\mu$, $A_{ab} = \epsilon^{acb}A^c_\mu$. We see that the constant field (84) requires the nonvanishing space-like current

$$J^a_\mu = \frac{2}{3}g^2A^2A^a_\mu \tag{86}$$

to be supported with. The classical field (84) obviously satisfies the current-conservation condition $\delta(\nabla^{ac}_k J^c_k(x) = 0$. In what follows we use the notation for the field invariant $\mathfrak{F} = (1/4)G^a_{\mu\nu}G^a_{\mu\nu}$. The normalization condition $\mathrm{d}^4\mathrm{Re}L/\mathrm{d}A^4|_{G_{(0)}} = -4g_r^2$ is imposed in an arbitrary normalization point $G^a_{\mu\nu} = G^a_{(0)\mu\nu}$ to fix the renormalized coupling constant g. Here $L = -\mathfrak{F} + \mathfrak{L}$ is the full and $\mathfrak{L}$ the effective Lagrangian, the tree Lagrangian being $-\mathfrak{F}$. According to Ref.[10] the calculation within one-gluon-one-ghost loop gives for the real part of the latter ($\mathfrak{F} \gg \mathfrak{F}_0$)

$$\mathrm{Re}\mathfrak{L} = -\mathfrak{F}\frac{25g^2}{16\pi^2} + \frac{3g^2}{16\pi^2}\mathfrak{F}\ln\frac{\mathfrak{F}}{\mathfrak{F}_0}. \tag{87}$$

(The principle of correspondence realizes differently from QED: radiative corrections contribute also into the part, linear in $\mathfrak{F}$, since the normalization point $\mathfrak{F}_0$ is not zero.)

It is seen that the Lagrangian (87) is a convex function of $\mathfrak{F}$, $\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = (3g^2/16\pi^2\mathfrak{F}) > 0$, throughout the whole magneticlike domain of validity $\mathfrak{F} \gg \mathfrak{F}_0$, unlike the Matinian-Savvidy and Kawati-Kokado Lagrangians considered in Subsections C, D. Consequently, no constant magnetic field is spontaneously produced. However, the presence of nonzero imaginary part of the Lagrangian of Ref. [10], $\mathrm{Im}\mathfrak{L} = -(12.15g^2/6\pi^2)\mathfrak{F}$, makes the theory unstable under creation of gluonic tachyons. Unlike the case of Subsection C, their spectra turn to zero in the zero-momentum point (see [10] for details), which explains, why no constant field is gained in the present case. As for condition (28), it is violated for $\mathfrak{F} > \mathfrak{F}\exp(22+16\pi^2/3g^2)$. Therefore the effective Lagrangian in the theory of Ref.[10] is closer to that of Euler-Heisenberg in what concerns its causal-unitarity properties: condition (31) is fulfilled for arbitrarily large magnetic field, while condition (28) is violated in the domain of exponentially large fields, which signifies the lack of asymptotic freedom in the both theories. (We cannot check conditions (27) and (29), since calculations with the second field-invariant kept different from zero are not available.)

IV. DISCUSSION

In the present paper, for establishing obligatory properties of the effective Lagrangian we exploited two general principles – unitarity and causality – taken in the special form of the requirements of nonnegativity of the residue (16) and of boundedness of the group velocity (14). We feel it necessary to confront this way of action with other approaches.

Usually, consequences of causality and unitarity are discussed referring to holomorphic properties of the polarization operator (or of the dielectric permittivity tensor) that follow from the retardation of the linear response and are expressed – after being supplemented by certain postulates concerning the high-frequency asymptotic conditions – as the Kramers-Kronig (once-subtracted) dispersion relations. Although the general proof of an analog of the Kramers-Kronig relation in a background field is lacking from the literature, for the magnetized vacuum the holomorphity of the polarization operator eigenvalues $\varkappa_a$ in a cut complex plane of the variable $(k_0^2 - k_3^2)$ was established within the one-loop approximation [11], [12], the probability of electron-positron pair creation by a photon making the cut

discontinuity. Nevertheless, as we could see in Section III A, this approximation includes appearance of negative-norm ghosts and tachyons in contradiction with causality and unitarity. Thus, the knowledge of the holomorphic properties is not enough to be sure that the causality and unitarity requirements have been exploited at full.

More specifically the causality is approached by referring to what is called "causal propagation". Here the Hadamard's method [26] of characteristic surface (the wave front), across which the first derivative of the propagating solution may undergo a discontinuity is used. The propagation is causal if the normal vector to the characteristic surface is time- or light-like. Once the coefficients in the differential equation responsible for the wave propagation are restricted in such a way as to meet this requirement, the wave front propagates exactly with the speed of light $c = 1$ [27] and should be equal to the phase velocity taken at infinite value of the frequency according to the Leontovich theorem [28]. (Note, however, that the infinite-frequency limit cannot be covered by any finite-order differential equation; on the contrary, when considering the general case of non-polynomial dispersion the Schwinger-Dyson set of equations should be taken seriously as integro-differential equations). Certain conditions obtained in this way that should be obeyed by the "structural function H", the knowing of which is equivalent to the effective Lagrangian, may be found among numerous relations in a scrupulous study of Jerzy Plebański. It seems, however, that inequalities (9.176) derived in his Lectures [21], relating to the general case $\mathfrak{F} \neq 0$, $\mathfrak{G} \neq 0$, and the subsequent formulae, relating to the null-field subcase, $\mathfrak{F} = \mathfrak{G} = 0$, need to be supplemented by consequences of some requirements intended to substitute for unitarity or positiveness of the energy, not exploited in [21], before/in-order-that a comparison with our conclusions might become possible. In the case of nontrivial dispersion, however, a coincidence is not even to be expected. The point is that the requirement that the wave front should not propagate faster than light is only a necessary, but not yet sufficient condition of the causal propagation: other signals should not be faster than light, either. It is widely recognized [29] that the group velocity is the speed of the wave packet at least where no anomalous dispersion is present, in which case the group velocity loses its interpretation as the wave packet speed and may exceed unity. An extension of the group velocity into the domain of anomalous dispersion that keeps it below the speed of light is also possible [5]. In Ref. [5] we also argued why the excess of the group velocity over the speed of light encountered in some problems with a violation of the Lorentz invariance should be viewed upon as a

serious discrepancy with the relativity principle, understood in this case as equivalence of a given reference frame, in which an external agent like a background field is also present, with another inertial frame, in which there is the same external agent, but Lorentz-boosted from the initial frame.

This is why we treat the group velocity criterion as the causality criterion in the present paper as well as in [5]. Previously the appeal to the group velocity has shown its fruitfulness in establishing the phenomenon of canalization of the photon energy along the external magnetic field [30], [11] and the capture of gamma-quanta by a strong nonhomogeneous magnetic field of a pulsar [31], [32]. As for the violation of the group velocity criterion for exponentially strong magnetic field discovered for the one-loop approximation in Section III, we admitted that the necessary value of the magnetic field cannot be achieved, because the magnetic field becomes unstable already at smaller values. Therefore, a magnetic field higher than that, for which the photon may become superluminal, is to be ruled out like people use to rule out perfectly elastic body, although in the latter case no mechanism that would ban its formation is considered.

On the other hand, the fulfillment of the Dominant Energy Condition (DEC) (44) implies that the causality is reassured, because when solving the Cauchy problem initial data have no influence on what occurs outside their light cone. (This is proved in [6] within General Relativity context.) We saw in Subection II D that the group-velocity criterion is equivalent to DEC in what concerns the consequences for the effective Lagrangian as a function of constant magnetic-like background field, although the implementation of DEC and WEC to the problem of elementary excitations over the magnetized vacuum undertaken in Subsection C of Section II has indicated, however, as we already discussed it in that subsection, that these two conditions together lead to somewhat weaker conclusions than the ones that followed in Subsection B from imposing the conditions of unitarity in the form (16) and causality in the form(14).

This work was supported by the Russian Foundation for Basic Research (Project No. 05-02-17217), as well as by the Israel Science Foundation of the Israel Academy of Sciences and Humanities.

References

[1] S. Weinberg, *The Quantum Theory of Fields*, (University Press, Cambridge, 2001).

[2] Irina Ya. Aref'eva, *AIPConf.Proc.* **957**, 297 (2007); arXiv:0710.3017

[3] M. Novello, S.E. Perez Bergliaffa and J. Salim, Phys.Rev. D **69**, 127301 (2004).

[4] I. A. Batalin and A.E.Shabad, Zh. Eksp.Teor.Fiz. **60**, 894 (1971) [Sov.Phys.JETP **33**, 483 (1971)].

[5] A. E. Shabad and V. V. Usov, Phys.Rev. D **81**, 125008 (2010).

[6] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (University Press, Cambrige, 1973).

[7] I.A. Batalin, S.G. Matinian and G.K. Savvidy, Yad. Fiz. **26**, 408 (1977).

[8] S. Kawati and A. Kokado, Phys. Rev. D **39**, 3612 (1989).

[9] A. Cabo and A.E. Shabad, in *Polarization effects in external gauge fields.* Ed. V.L. Ginzburg, Proc. P.N. Lebedev Phys. Inst. **192**, 151 (Nauka, Moscow, 1988) in Russian.

[10] A. Cabo and A.E. Shabad, Acta Phys. Polonica B, **17**, 591 (1986); in *Group-Theoretical Methods in Physics.* Ed. M.A. Markov, Proc. Second 1982 Zvenigorod Sem. **2**,153 (Chur, London *etc.*, Harwood Acad. Publ., 1987).

[11] A. E. Shabad, Ann. Phys. **90**, 166 (1975).

[12] A.E. Shabad, *Polarization of the Vacuum and a Quantum Relativistic Gas in an External Field* (Nova Science Publishers, New York, 1991). See also in *Polarization effects in external gauge fields.* Ed. V.L. Ginzburg, Proc. P.N. Lebedev Phys. Inst. **192**, 5 (Nauka, Moscow, 1988) in Russian

[13] A. E. Shabad and V. V. Usov, arXiv:0911.0640 [hep-th]

[14] The Lagrangian equivalent to (47) was previously derived in A. Di Piazza and G. Calucci, Phys.Rev. D **99**, 123006 (2002)

[15] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936); V. Weiskopf, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **14**, 6 (1936); J. Schwinger, Phys. Rev. **82**, 664 (1951); V.B. Berestetsky, E.M. Lifshits, and L.P. Pitayevsky, *Quantum Electrodinamics* (Nauka, Moscow, 1989; Pergamon Press Oxford, New York, 1982).

[16] It is associated with logarithmically growing term in the Euler-Heisenberg Lagrangian as discussed by V.I. Ritus, Zh.Eksp.Teor.Fiz. **69**, 1517 (1975); **73**, 807 (1977) [Sov.Phys. JETP **42**, 774 (1975); **46**, 423 (1977)].

[17] The presence of the linear term in the polarization operator was established in V. V. Skobelev, Izv. Vyssh. Uchebn. Zaved. Fiz. No**10**, 142 (1975); D.B. Melrose and R.J. Stoneham, Nuovo Cimento **A 32**, 435 (1976); A. E. Shabad, Kratkie Soobshchenia po Fizike [Sov. Phys. - Lebedev Inst. Reps.] **3**, 13 (1976); see also [12] and J.S. Heyl and L. Hernquist, J.Phys.A **30**, 6485 (1997).

[18] A.E. Shabad and V.V. Usov, Phys.Rev.Lett. **96**, 180401 (2006); A.E. Shabad and V.V. Usov, Phys.Rev. D **73**, 125021 (2006).

[19] M. Born and L. Infeld, Proc. Roy. Soc. A **144**, 425 (1934).

[20] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **163**, 123 (1985).

[21] Jerzy Plebański, *Lectures on Nonlinear Electrodynamics* (Nordita, Copenhagen, 1970).

[22] A.E. Shabad and V.V. Usov, Phys.Rev. D **77**, 025001 (2008); Phys. Rev. Lett.**98**, 180403 (2007); arXiv:0801.0115[hep-th].

[23] G.K. Savvidy, Phys.Lett. **71B**, 133 (1977).

[24] N.K. Nielsen and P. Olesen, Nucl.Phys. B **144**, 376 (1978).

[25] A. Cabo, Fortsch. der Phys. **29**, 495 (1981).

[26] R. Courant and D. Gilbert *Methods of Mathematical Physics. Vol.2* (Interscience, New York, 1962).

[27] L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New Yourk, 1960).

[28] See the 1937 talk of M.A. Leontovich, also L.I. Mandelshtam's comments to it, in collection: L.I. Mandelstam *Lectures in Optics, Relativity, and Quantum Mechanics,* Ed. S.M. Rytov (Nauka, Moscow, 1972) in Russian.

[29] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Oxford, 1968)

[30] A.E. Shabad, Lett. Nuovo Cimento **2**, 457 (1972).

[31] A.E. Shabad and V.V. Usov, Nature **295**, 215 (1982).

[32] A. E. Shabad and V. V. Usov, Astrophys. Space Sci. **117**, 309 (1985), **128**, 377 (1986).